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Effective high-order approximations of layered coatings and linings of anisotropic elastic, viscoelastic and nematic materials☆

D.D. Zakharov

Moscow, Russia

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ABSTRACT

Beginning at low frequencies, asymptotically exact models of anisotropic coatings and linings with a small ratio of the half-thickness to the longitudinal deformation scale are constructed. The requirement on the conditions of contact with the substrate, where at least one of the boundary conditions must contain a displacement component of the strain in explicit form is "non-classical" here. The action of the coating/lining on a thicker body is approximated by impedance boundary conditions at the interface. The error of the model is reduced to the third order for layered packets and to the sixth order for a single layer. The physical limit of the applicability is the frequency of the first quasi-resonance in the corresponding deformed system. A comparison with the propagator matrix and numerical testing for partial waves shows satisfactory accuracy, comparable with the accuracy of the theory of classical plates of similar order. The results can be used in contact problems and for rapid algorithms for calculating the spectrum of the eigenwaves in half-spaces and thick layered plates with any number of coatings and linings. An extension to the case of viscoelastic materials and nematic elastomers is given.

An approximate description of the action of a thin layer using effective boundary conditions at the interface of media has been successfully used to calculate electromagnetic fields for more than half a century.[1,2] This idea is applied here to problems of the dynamics of solids. Approximate models of single-layer coatings and linings were constructed earlier to solve contact problems[3-5] and to model adhesion properties[5-10] using both asymptotic and numerical methods. Asymptotic integration was used to construct approximate relations in isotropic coatings and linings,[11,12] and also in thermoelastic anisotropic double-layer plates, pressed by rigid faces.[13,14] All these models were quasi-static and had a comparatively low order of accuracy. These approximations are of interest for problems of dynamics, where, despite developed matrix methods of calculating wave propagation,[15-19] the problem of carrying out an effective calculation of the spectra of the eigenwaves in layered solids remains a pressing one. The main difficulties arise for composite structural components with complex properties - a large number of layers, anisotropy of the materials and viscosity, and "non-traditional" behaviour of new materials, which affects the speed of the algorithms and their stability at high frequencies.

The construction of analogs of the impedance boundary conditions (IBC) for one elastic and one electroelastic layer was limited to extremely long waves,[20-28] which does not enable many dynamic effects to be correctly described. A typical approach reduces to the asymptotic expansion of the propagator matrices in a power series in the wave number, which is extremely cumbersome for subsequent refinements. The use of Padé approximations and Magnus expansion[29,30] enables an analog of the IBC of higher orders to be obtained, but used components of different orders and does not guarantee a uniform asymptotic error in approximating the stress-strain state of the coatings or linings.

Below we use asymptotic integration to construct the IBC for layered coatings and linings with an asymptotic error of up to the third order and sixth order, when there is one layer. A general form of elastic anisotropy is admitted in the layer materials. The case of dissipative materials, including the case of liquid-crystalline (nematic) elastomers[31-34] is considered separately.

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E-mail address: dd_zakh@mail.ru.

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1. Asymptotic description of the internal stress-strain state of a packet with constrained faces

Consider a packet of N anisotropic elastic layers, where the j-th layer has a thickness H_j: $Z \equiv X_3, Z_j \le Z \le Z_{j+1}$ ($j = 1,2,\ldots, N$) occupying the region $-\infty < X_1, X_2 < \infty$ in its plan. We will denote its stiffness matrix by $\mathbf{G}_j = \|g^j_{pg}\|$ and the density by ρ_j. In obvious cases we will omit the index of the layer. We will introduce the following notation: for longitudinal and transverse displacements $\mathbf{U} \equiv (U_1,U_2)^T$ and $W \equiv U_3$, for the stresses σ_{pq} and the strains $\varepsilon_{pp} = \partial_p U_p$, $\gamma_{pq} = \partial_p U_q + \partial_q U_q$ related by Hooke's law. We will assume that the conditions of complete contact at the layer interfaces is satisfied. We will specify the conditions on the faces $Z^- = Z_1$, $Z^+ = Z_{N+1}$ ($Z^+ - Z^- = 2h$) in more detail later, but we will have in mind that the inhomogeneous boundary conditions are non-classical and at least for one of the faces must contain at least one component of the displacements U_p^- or U_p^+ in explicit form, i.e., the problem is completely analogous to that considered earlier.[11–14]

We will investigate the internal stress-strain state in the layers, which satisfy the equations of motion (T is the time)

$$\rho \partial_T^2 U_p = \sum_q \partial_q \sigma_{pq} \tag{1.1}$$

and we will obtain relations between the values of the stresses $\sigma_{p3}^{\mp}$ and strains $U_p^{\mp}$ on the faces $Z = Z^{\mp}$. We will assume the ratio of the half-thickness of the packet h to the characteristic scale of the process L in the longitudinal direction to be a small parameter: $\varepsilon = h/L \ll 1$ and we will proceed to dimensionless variables

$$\mathbf{x} = \mathbf{X}/L \quad (\mathbf{X} = (X_1, X_2)), \quad z = Z/h, \quad t = T/T_0 \quad \left(T_0 = \varepsilon^{\tau-1} L/c_0,\ c_0 \equiv \sqrt{E_0/\rho_0}\right)$$

Here ρ_0 is the density scale, E_0 is the characteristic modulus of elasticity (for example, the smallest of the shear moduli in the layers) and τ is the coefficient of dynamics, characterizing the timescale.

We will seek the strains and stresses in the form of formal asymptotic series

$$\mathbf{U} = h\varepsilon^{\lambda}\left\{\mathbf{u}^0 + \varepsilon \mathbf{u}^1 + \ldots\right\}, \quad W = h\varepsilon^{\lambda}\left\{w^0 + \varepsilon w^1 + \ldots\right\}, \quad \sigma_{pq} = E_0 \varepsilon^{\lambda}\left\{\sigma^0_{pq} + \varepsilon \sigma^1_{pq} + \ldots\right\} \tag{1.2}$$

determining the variability factors λ and τ in subsequent considerations. It can be verified that the relations between the terms of the series lose their recurrence when $\tau > 1$, so that we can choose the maximum possible value $\tau = 1$. Then the scale T_0 corresponds to the least period of shear waves in the packet. In the classical theory of Plates[11,35,36] another parameter $\eta = h/L_0 \ll 1$ often appears, where L_0 is the characteristic geometrical dimension of the body and the deformation scale $L = L_0 \eta^Q$ ($0 < Q < 1$). However, when considering the internal stress-strain state the problem is reduced to expansion in powers of $\eta^{1-Q} = \varepsilon$.

We will normalize all the stiffnesses to E_0, the densities to ρ_0 and the thicknesses and displacements to h. We will introduce the following dimensionless vectors and matrix operators

$$\mathbf{d}_j = \begin{bmatrix} w \\ u_2 \\ u_1 \end{bmatrix}_j, \quad \mathbf{t}_{zj} = \begin{bmatrix} \sigma_{zz} \\ \sigma_{2z} \\ \sigma_{1z} \end{bmatrix}_j, \quad \mathbf{t}_{xj} = \begin{bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{22} \end{bmatrix}_j, \quad \sigma_j = \begin{bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{22} \\ \sigma_{2z} \\ \sigma_{1z} \end{bmatrix}_j, \quad \mathbf{D} = \begin{bmatrix} 0 & 0 & 0 & \partial_2 & \partial_1 \\ 0 & \partial_1 & \partial_2 & 0 & 0 \\ \partial_1 & \partial_2 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{G}_{0j} = \mathbf{G}_j\begin{pmatrix} 345 \\ 345 \end{pmatrix}, \quad \mathbf{G}_{*j} = \mathbf{G}_j\begin{pmatrix} 162 \\ 345 \end{pmatrix}, \quad \mathbf{G}_{\perp j} = \mathbf{G}_j\begin{pmatrix} 16245 \\ 162 \end{pmatrix}, \quad \mathbf{G}_{\| j} = \mathbf{G}_j\begin{pmatrix} 16245 \\ 345 \end{pmatrix}$$

$$\mathbf{D}_{1j} = \mathbf{D}\mathbf{G}_{\| j} + \mathbf{G}_{\| j}^T \mathbf{D}^T, \quad \mathbf{D}_{2j} = \mathbf{D}\mathbf{G}_j\begin{pmatrix} 16245 \\ 16245 \end{pmatrix}\mathbf{D}^T, \quad \mathbf{N}_j = \mathbf{D}\mathbf{G}_{\| j}\mathbf{G}_{0j}^{-1}$$

$$\mathbf{A}_j = \frac{\rho_j}{\rho_0}\partial_t^2 - \mathbf{D}_{2j}, \quad \mathbf{B}_j = \mathbf{A}_j + \mathbf{D}_{1j}\mathbf{G}_{0j}^{-1}\mathbf{D}_{1j}$$

and the following additional notation, which will be convenient for calculating quantities at the interfaces

$$\mathbf{d}^- = \mathbf{d}_1\big|_{z=z^-}, \quad \mathbf{d}^+ = \mathbf{d}_N\big|_{z=z^+}, \quad \mathbf{t}_z^- = \mathbf{t}_{z1}\big|_{z=z^-}, \quad \mathbf{t}_z^+ = \mathbf{t}_{zN}\big|_{z=z^+},$$

$$\mathbf{d}_{\pm} = \left(\mathbf{d}^+ \pm \mathbf{d}^-\right)/2, \quad \mathbf{t}_{\pm} = \left(\mathbf{t}_z^+ \pm \mathbf{t}_z^-\right)/2$$

$$z_j^- = z_j, \quad z_j^+ = z_{j+1}, \quad z_j^0 = \frac{1}{2}\left(z_j^+ + z_j^-\right), \quad z^{\mp} \equiv z_1, z_{N+1}, \quad \sum\nolimits_- \equiv \sum_{k=1}^{j-1}, \quad \sum\nolimits_+ \equiv \sum_{k=j+1}^{N}$$

The minors $\mathbf{G}_{0j}, \mathbf{G}_{*j}, \mathbf{G}_{\perp j}$ and $\mathbf{G}_{\| j}$ are obtained from the elements of the matrix $\mathbf{G}_j$, situated at the intersection of the row numbers indicated (the superscripts) and the column numbers indicated (the subscripts). By virtue of Eqs (1.1) and Hooke's law, the dimensionless s-th terms in expansions (1.2) satisfy the relations

$$\mathbf{t}_{zj}^s = \mathbf{G}_{\| j}^T \mathbf{D}^T \mathbf{d}_j^{s-1} + \mathbf{G}_{0j}\partial_z \mathbf{d}_j^s, \quad \mathbf{t}_{xj}^s = \mathbf{G}_{\perp j}^T \mathbf{D}^T \mathbf{d}_j^{s-1} + \mathbf{G}_{*j}\partial_z \mathbf{d}_j^s \tag{1.3}$$

$$\partial_z \mathbf{t}_{zj}^s = -\mathbf{D}\sigma_j^{s-1} + \frac{\rho_j}{\rho_0}\partial_t^2 \mathbf{d}_j^{s-2}, \quad \partial_z^2 \mathbf{d}_j^s = \mathbf{G}_{0j}^{-1}\left(-\mathbf{D}_{1j}\partial_z \mathbf{d}_j^{s-1} + \mathbf{A}_j \mathbf{d}_j^{s-2}\right) \tag{1.4}$$

the interface conditions and the conditions on the faces ($\delta_0^{\lambda+s+1}$ is the Kronecker delta)

$$z = z_{j+1}: \mathbf{t}_{zj}^s = \mathbf{t}_{zj+1}^s, \quad \mathbf{d}_j^s = \mathbf{d}_{j+1}^s; \quad j = 1, 2, \ldots, N-1 \tag{1.5}$$

$$z = z^{\mp}: \mathbf{t}_z^{s,\mp} = \mathbf{t}_z^{\mp}\delta_0^{\lambda+s}, \quad \mathbf{d}^{s,\mp} = \mathbf{d}^{\mp}\delta_0^{\lambda+s+1} \tag{1.6}$$

Hence, the strains and stresses are found from the recurrence formulae (1.3)–(1.6) by successive integration over the thickness for $s = 0$, 1, ...

2. Boundary relations for $s = 0, 1, 2$

We will derive relations which enable us to establish the stress-strain state from the values on the faces. For $s = 0$ we have

$$\mathbf{d}_j^0 = \mathbf{d}_{0j}^0 + z\mathbf{d}_{1j}^0; \quad \mathbf{d}_{0j}^0 = \mathbf{d}_+^0 - \left(\mathbf{G}_{+j}^{-1} - \mathbf{G}_{-j}^{-1} + z_j^0\mathbf{G}_{0j}^{-1}\right)\mathbf{t}_+^0, \quad \mathbf{d}_{1j}^0 = \mathbf{G}_{0j}^{-1}\mathbf{t}_+^0$$

$$\mathbf{t}_{zj}^0 = \mathbf{t}_+^0, \quad \mathbf{t}_{xj}^0 = \mathbf{G}_{*j}\mathbf{G}_{0j}^{-1}\mathbf{t}_+^0 \left(\mathbf{G}_0^{-1} \equiv \frac{1}{2}\sum_j h_j \mathbf{G}_{0j}^{-1},\ \mathbf{G}_{\pm j}^{-1} \equiv \frac{1}{2}\sum\nolimits_{\pm} h_k \mathbf{G}_{0k}^{-1}\right)$$

whence we have the following equalities

$$\mathbf{d}_-^0 = \mathbf{G}_0^{-1}\mathbf{t}_+^0, \quad \mathbf{t}_-^0 = 0 \tag{2.1}$$

For $s = 1$

$$\mathbf{d}_j^1 = \mathbf{d}_{0j}^1 + z\mathbf{d}_{1j}^1 + \frac{z^2}{2}\mathbf{d}_{2j}^1$$

$$\begin{aligned}\mathbf{d}_{0j}^1 = \mathbf{d}_+^1 - &\left\{\mathbf{G}_{+j}^{-1} - \mathbf{G}_{-j}^{-1} + z_j^0\mathbf{G}_{0j}^{-1}\right\}\mathbf{t}_+^1 + \left\{\mathbf{N}_{+j}^T - \mathbf{N}_{-j}^T + z_j^0\mathbf{N}_j^T\right\}\mathbf{d}_+^0 - \\ &\left\{\mathbf{C}_j^+ - \mathbf{C}_j^- + z_j^0\mathbf{C}_j + \tfrac{1}{2}z_j^+ z_j^- \mathbf{G}_{0j}^{-1}\mathbf{D}_{1j}\mathbf{G}_{0j}^{-1}\right\}\mathbf{t}_+^0\end{aligned}$$

$$\mathbf{d}_{1j}^1 = \mathbf{G}_{0j}^{-1}\mathbf{t}_+^1 - \mathbf{N}_j^T\mathbf{d}_+^0 + \left\{\mathbf{C}_j + z_j^0\mathbf{G}_{0j}^{-1}\mathbf{D}_{1j}\mathbf{G}_{0j}^{-1}\right\}\mathbf{t}_+^0, \quad \mathbf{d}_{2j}^1 = -\mathbf{G}_{0j}^{-1}\mathbf{D}_{1j}\mathbf{G}_{0j}^{-1}\mathbf{t}_+^0$$

$$\left(\mathbf{C}_j = \mathbf{N}_j^T\left(\mathbf{G}_{+j}^{-1} - \mathbf{G}_{-j}^{-1}\right) + \mathbf{G}_{0j}^{-1}\left(\mathbf{N}_j^+ - \mathbf{N}_j^-\right),\ \mathbf{C}_j^{\pm} = \frac{1}{2}\sum\nolimits_{\pm} h_k\mathbf{C}_k,\ \mathbf{C} = \frac{1}{2}\sum_j h_j\mathbf{C}_j\right)$$

$$\mathbf{t}_{zj}^1 = \mathbf{t}_{0zj}^1 + z\mathbf{t}_{1zj}^1: \mathbf{t}_{1zj}^1 = -\mathbf{N}_j\mathbf{G}_{0j}\mathbf{d}_{1j}^0 = -\mathbf{N}_j\mathbf{t}_+^0, \quad \mathbf{t}_{0zj}^1 = \mathbf{t}_+^1 + \left\{\mathbf{N}_j^+ - \mathbf{N}_j^- + z_j^0\mathbf{N}_j\right\}\mathbf{t}_+^0$$

$$\left(\mathbf{N} \equiv \frac{1}{2}\sum_j h_j\mathbf{N}_j,\ \mathbf{N}_j^{\pm} \equiv \frac{1}{2}\sum\nolimits_{\pm} h_k\mathbf{N}_k\right)$$

and the following relations hold

$$\mathbf{d}_-^1 = \mathbf{G}_0^{-1}\mathbf{t}_+^1 - \mathbf{N}^T\mathbf{d}_+^0 + \mathbf{C}\mathbf{t}_+^0, \quad \mathbf{t}_-^1 = 0 - \mathbf{N}\mathbf{t}_0^+ \tag{2.2}$$

Equalities (2.1) and (2.2) contain no time derivatives, i.e., they are quasi-static. The wave operator $\mathbf{A}_j$ appears when $s = 2$, which leads to expressions for the displacements

$$\mathbf{d}_j^2 = \mathbf{d}_{0j}^2 + z\mathbf{d}_{1j}^2 + \frac{z^2}{2!}\mathbf{d}_{2j}^2 + \frac{z^3}{3!}\mathbf{d}_{3j}^2$$

$$\begin{aligned}\mathbf{d}_{0j}^2 = \mathbf{d}_+^2 - &\left\{\mathbf{G}_{+j}^{-1} - \mathbf{G}_{-j}^{-1} + z_j^0\mathbf{G}_{0j}^{-1}\right\}\mathbf{t}_+^2 + \left\{\mathbf{N}_{+j} - \mathbf{N}_{-j} + z_j^0\mathbf{N}_j\right\}^T\mathbf{d}_+^1 - \\ &\left\{\mathbf{C}_j^+ - \mathbf{C}_j^- + z_j^0\mathbf{C}_j + \frac{1}{2}z_j^+ z_j^- \mathbf{G}_{0j}^{-1}\mathbf{D}_{1j}\mathbf{G}_{0j}^{-1}\right\}\mathbf{t}_+^1 + \\ &\left\{\mathbf{E}_j^+ - \mathbf{E}_j^- + z_j^0\mathbf{E}_j + \frac{1}{2}z_j^+ z_j^-\left(\mathbf{A}_j + \mathbf{N}_j\mathbf{D}_{1j}\right)\mathbf{G}_{0j}^{-1}\right\}^T\mathbf{d}_+^0 - \\ &\left\{\mathbf{T}_j^+ - \mathbf{T}_j^- + z_j^0\mathbf{T}_j + \frac{1}{2}z_j^+ z_j^-\left(\mathbf{M}_j + \frac{1}{3}z_j^0\mathbf{G}_{0j}^{-1}\mathbf{B}_j\mathbf{G}_{0j}^{-1}\right)\right\}\mathbf{t}_+^0\end{aligned}$$

$$\mathbf{d}_{1j}^2 = \mathbf{G}_{0j}^{-1}\mathbf{t}_+^2 - \mathbf{N}_j^T\mathbf{d}_+^1 + \left\{\mathbf{C}_j + z_j^0\mathbf{G}_{0j}^{-1}\mathbf{D}_{1j}\mathbf{G}_{0j}^{-1}\right\}\mathbf{t}_+^1 -$$
$$\left\{\mathbf{E}_j + z_j^0(\mathbf{A}_j + \mathbf{N}_j\mathbf{D}_{1j})\mathbf{G}_{0j}^{-1}\right\}^T\mathbf{d}_+^0 + \left\{\mathbf{H}_j + z_j^0\mathbf{M}_j + \tfrac{1}{2}z_j^+z_j^-\mathbf{G}_{0j}^{-1}\mathbf{B}_j\mathbf{G}_{0j}^{-1}\right\}\mathbf{t}_+^0$$

$$\mathbf{d}_{2j}^2 = \mathbf{G}_{0j}^{-1}(-\mathbf{D}_{1j}\mathbf{G}_{0j}^{-1}\mathbf{t}_+^1 + [\mathbf{A}_j + \mathbf{D}_{1j}\mathbf{N}_j^T]\mathbf{d}_+^0 - [\mathbf{D}_{1j}\mathbf{C}_j + \mathbf{A}_j\{\mathbf{G}_{+j}^{-1} - \mathbf{G}_{-j}^{-1}\} + z_j^0\mathbf{B}_j\mathbf{G}_{0j}^{-1}]\mathbf{t}_+^0)$$

$$\mathbf{d}_{3j}^2 = \mathbf{G}_{0j}^{-1}\mathbf{B}_j\mathbf{G}_{0j}^{-1}\mathbf{t}_+^0$$

and the stresses

$$\mathbf{t}_{zj}^2 = \mathbf{t}_{0zj}^2 + z\mathbf{t}_{1zj}^2 + \frac{z^2}{2}\mathbf{t}_{2zj}^2$$

$$\mathbf{t}_{0zj}^2 = \mathbf{t}_+^2 + \left\{\mathbf{N}_j^+ - \mathbf{N}_j^- + z_j^0\mathbf{N}_j\right\}\mathbf{t}_+^1 - \left\{\mathbf{L}_j^+ - \mathbf{L}_j^- + z_j^0\mathbf{L}_j\right\}\mathbf{d}_+^0 +$$
$$\left\{\mathbf{F}_j^+ - \mathbf{F}_j^- + z_j^0\mathbf{F}_j + \frac{1}{2}z_j^+z_j^-(\mathbf{A}_j + \mathbf{N}_j\mathbf{D}_{1j})\mathbf{G}_{0j}^{-1}\right\}\mathbf{t}_+^0$$

$$\mathbf{t}_{1zj}^2 = -\mathbf{N}_j\mathbf{t}_+^1 + \mathbf{L}_j\mathbf{d}_+^0 - \left[\mathbf{F}_j + z_j^0(\mathbf{A}_j + \mathbf{N}_j\mathbf{D}_{1j})\mathbf{G}_{0j}^{-1}\right]\mathbf{t}_+^0, \quad \mathbf{t}_{2zj}^2 = (\mathbf{A}_j + \mathbf{N}_j\mathbf{D}_{1j})\mathbf{G}_{0j}^{-1}\mathbf{t}_+^0$$

with operators

$$\mathbf{L}_j = \mathbf{A}_j + \mathbf{N}_j\mathbf{G}_{0j}\mathbf{N}_j^T, \quad \mathbf{F}_j = \mathbf{N}_j\left(\mathbf{N}_j^+ - \mathbf{N}_j^-\right) + \mathbf{L}_j\left(\mathbf{G}_{+j}^{-1} - \mathbf{G}_{-j}^{-1}\right), \quad \mathbf{L}_2 = \mathbf{L}\mathbf{G}_0^{-1}$$

$$\mathbf{L} = \frac{1}{2}\sum_j h_j\mathbf{L}_j, \quad \mathbf{L}_j^\pm = \frac{1}{2}\sum\nolimits_\pm h_k\mathbf{L}_k \ (\mathbf{L} \leftrightarrow \mathbf{F})$$

$$\mathbf{E}_j^T = \mathbf{G}_{0j}^{-1}\left\{\mathbf{L}_j^+ - \mathbf{L}_j^-\right\} + \mathbf{N}_j^T\left\{\mathbf{N}_{+j}^T - \mathbf{N}_{-j}^T\right\},$$

$$\mathbf{H}_j = \mathbf{G}_{0j}^{-1}\left(\mathbf{F}_j^+ - \mathbf{F}_j^-\right) + \mathbf{N}_j^T\left(\mathbf{C}_j^+ - \mathbf{C}_j^-\right), \quad \mathbf{M}_j = \mathbf{G}_{0j}^{-1}\mathbf{F}_j + \mathbf{N}_j^T\mathbf{C}_j$$

$$\mathbf{E} = \frac{1}{2}\sum_j h_j\mathbf{E}_j, \quad \mathbf{E}_j^\pm = \frac{1}{2}\sum\nolimits_\pm h_k\mathbf{E}_k \ (\mathbf{E},\mathbf{H},\mathbf{M}), \quad \mathbf{T}_j = \mathbf{H}_j - \frac{1}{12}h_j^2\mathbf{G}_{0j}^{-1}\mathbf{B}_j\mathbf{G}_{0j}^{-1}, \quad \mathbf{T} = \frac{1}{2}\sum_j h_j\mathbf{T}_j$$

On the faces we obtain the relations

$$\mathbf{d}_-^2 = \mathbf{G}_0^{-1}\mathbf{t}_+^2 - \mathbf{N}^T\mathbf{d}_+^1 + \mathbf{C}\mathbf{t}_+^1 - \mathbf{E}^T\mathbf{d}_+^0 + \mathbf{T}\mathbf{t}_+^0, \quad \mathbf{t}_-^2 = 0 - \mathbf{N}\mathbf{t}_+^1 + \mathbf{L}\mathbf{d}_+^0 - \mathbf{F}\mathbf{t}_+^0 \tag{2.3}$$

Summing over s = 0, 1, 2, we arrive at a dimensional form of the symmetrical boundary relations (2.1)–(2.3)

$$\mathbf{d}_- = \left(\mathbf{G}_0^{-1} + \mathbf{C} + \mathbf{T}\right)\mathbf{t}_+ - (0 - \mathbf{N} + \mathbf{E})^T\mathbf{d}_+, \quad \mathbf{t}_- = (0 - \mathbf{N} - \mathbf{F})\mathbf{t}_+ + (0 + 0 + \mathbf{L})\mathbf{d}_+ \tag{2.4}$$

with a relative asymptotic error O(ε^3) with respect to the order of neglected terms. With the same accuracy, we obtain from equalities (2.1)–(2.3) the following equivalent asymmetrical boundary relations

$$\mathbf{t}_- + [0 + \mathbf{N} + (\mathbf{F} \pm \mathbf{L}_2)]\mathbf{t}_+ = (0 + 0 + \mathbf{L})\mathbf{d}^\pm \tag{2.5}$$

$$\mathbf{d}_- + [0 + \mathbf{N} + (\mathbf{E} \pm \mathbf{L}_2)]^T\mathbf{d}_+ = \left[\mathbf{G}_0^{-1} + \left(\mathbf{C} \pm \mathbf{G}_0^{-1}\mathbf{N}\right) + \left(\mathbf{T} + \mathbf{G}_0^{-1}\mathbf{N}^2 \pm \mathbf{G}_0^{-1}\mathbf{F} \pm \mathbf{C}\mathbf{N}\right)\right]\mathbf{t}_z^\pm \tag{2.6}$$

No specific values of λ have yet been used. They can, in principle, be determined from additional information. For example, knowing that when $\varepsilon \to +0$ in formulae (1.6) $\mathbf{d}^\mp = O(1)$, we obtain $\lambda = -1$. This is not essential for a further discussion since relations (2.4)–(2.6) for coatings and linings hold for any order of λ, produced when thick deformed bodies with thin coatings and linings interact.

3. Impedance boundary conditions for layered coatings and linings

We will now consider the contact between a thin packet and thicker elastic bodies - half-spaces or plates. We will assume that the characteristic scale in a thick body is comparable with the value of L, and the values of the densities and other parameters in the body are comparable with the values for the packet (which can also be achieved for fairly small ε). The scaling of the variables in the thick body is then only changed for the transverse coordinate $z = Z/L$, i.e., scaling does not change the form of the elasticity relations and the equations of motion. Nevertheless, the displacements and strains in the thicker solid may be represented in the form of asymptotic series in powers of ε by writing the contact conditions for the corresponding terms in the thin packet; but the equations for the s-th terms of the series in the thick solid are similar to the usual relations and the equations of the theory of elasticity, and are not by themselves recurrence relations.

Hence, the specification, for example, of the ideal condition of contact with one or two more thick solids in the form of continuity of the displacements and the transverse stresses does not distort the asymptotic integration procedure described in Sections 1 and 2. Consequently, in the impedance boundary conditions (2.4)–(2.6) the quantities $\mathbf{t}_z^{\pm}$ and $\mathbf{d}^{\pm}$ may denote displacements and stresses in elastic substrates (if there are two), or one of them may correspond to an elastic substrate, while the other is the specified boundary condition on the face. The asymptotic accuracy of the impedance boundary conditions then remains as before. Depending on the type boundary-value problem set up for an approximate description of the action of the lining (coating), one can choose both equalities (2.4) or one of the equalities (2.5), (2.6), and then find the solution of the three-dimensional equations for the thick solid. One can similarly derive relations for a slipping contact with slipping, friction, etc.

The impedance boundary conditions obtained are not independent equations; they are relations between boundary values which do not have additional degrees of freedom. As $h \to 0$ impedance boundary conditions (2.4)–(2.6) give the usual conditions of continuity of the displacements and stresses on the interface, or equality of the displacements or stresses in a thick solid with specified perturbations on the face. The fact that the impedance boundary conditions for a packet were constructed for inhomogeneous boundary conditions on the faces does not limit the generality. In particular, they are completely applicable to the problem of determining the eigenwaves and spectra in a thick solid with coatings/linings in a reasonable frequency band ω. Here the thicker solid is the source of a perturbation for the packet (in displacements, i.e., simultaneously it acts as a limiter). The physical basis for the impedance boundary conditions to be applicable is the difference in scales: in the scale of the coating or the lining it is necessary to remain in the low-frequency region $\omega h/c_0 \ll 1$, but in the scale of the thick solid this is not necessary, i.e., for its half-thickness h_* and the least velocity c_* is sufficient to satisfy the inequality $\omega h/c_0 \ll \omega h_*/c_*$.

In the limit situation for a layer, inhomogeneous over the thickness, with a rigid limiter, as is well known, there are no fundamental modes in the low-frequency range. In this case, in the asymptotic model of the layer there are also no independent differential equations. A qualitatively different situation can be expected for contrast media, where the limiting case of a stiff layer in a soft medium leads to the equations of motion of a thin plate (see Refs. 37–41 for "classical" contrast layered plates), which is outside the scope of this paper.

4. Impedance boundary conditions for a single layer

For one layer the impedance boundary conditions can be simplified considerably, since $j=1$, $\sum_{\mp}=0$ and, continuing the iterations, one can obtain relations for orders $s=3, 4, 5$. Omitting the intermediate consider, we will present dimensional relations with an asymptotic error $O(\varepsilon^6)$ for symmetrical impedance boundary conditions

$$\mathbf{t}_- = \left(0 - \mathbf{N}_j\mathbf{G}_{0j} + \frac{h^2}{3}\mathbf{A}_{3j} - \frac{h^4}{5}\mathbf{A}_{5j}\right)\mathbf{d}_- + \left(h\mathbf{A}_j - \frac{h^3}{3}\mathbf{A}_{4j}\right)\mathbf{d}_+$$

$$h\mathbf{t}_+ = \left(\mathbf{G}_{0j} + \frac{h^2}{3}\mathbf{B}_j - \frac{h^4}{5}\mathbf{B}_{4j}\right)\mathbf{d}_- + \left(h\mathbf{G}_{0j}\mathbf{N}_j^T - \frac{h^3}{3}\mathbf{B}_{3j} + \frac{h^5}{6}\mathbf{B}_{5j}\right)\mathbf{d}_+ \quad (4.1)$$

and asymmetrical impedance boundary conditions

$$\mathbf{t}_- + h\left(\mathbf{N}_j \pm h\mathbf{L}_{2j} + h^2\mathbf{S}_{3j} \pm h^3\mathbf{R}_{4j} + h^4\mathbf{P}_{5j}\right)\mathbf{t}_+ = h\left(0 + \mathbf{L}_j \pm h\mathbf{L}_j\mathbf{N}_j^T + h^2\mathbf{M}_{4j} \pm h^3\mathbf{Q}_{5j}\right)\mathbf{d}^{\pm} \quad (4.2)$$

$$\mathbf{d}_- + h\left(\mathbf{N}_j \pm h\mathbf{L}_{2j} + h^2\mathbf{S}_{3j} \pm h^3\mathbf{R}_{4j} + h^4\mathbf{P}_{5j}\right)^T\mathbf{d}_+ =$$
$$h\mathbf{G}_{0j}^{-1}\left(1 \pm h\mathbf{N}_j + h^2\mathbf{N}_{2j} \pm h^3\mathbf{R}_{3j} + h^4\mathbf{T}_{4j} \pm h^5\mathbf{Y}_{5j}\right)\mathbf{t}_z^{\pm} \quad (4.3)$$

with the corresponding matrix differential operators

$$\mathbf{A}_{3j} \equiv \mathbf{A}_j\mathbf{G}_{0j}^{-1}\mathbf{D}_{1j}, \quad \mathbf{A}_{4j} \equiv \mathbf{A}_j\mathbf{G}_{0j}^{-1}\mathbf{A}_j, \quad \mathbf{A}_{5j} \equiv \frac{2}{3}\mathbf{A}_{4j}\mathbf{G}_{0j}^{-1}\mathbf{D}_{1j} + \frac{1}{4}\mathbf{A}_{3j}\mathbf{G}_{0j}^{-1}\mathbf{B}_j$$

$$\mathbf{B}_{3j} \equiv \mathbf{A}_{3j}^T, \quad \mathbf{B}_{4j} \equiv \frac{2}{3}\mathbf{B}_{3j}\mathbf{G}_{0j}^{-1}\mathbf{D}_{1j} + \frac{1}{4}\mathbf{B}_j\mathbf{G}_{0j}^{-1}\mathbf{B}_j, \quad \mathbf{B}_{5j} \equiv \frac{4}{5}\mathbf{B}_{3j}\mathbf{G}_{0j}^{-1}\mathbf{A}_j + \frac{2}{15}\mathbf{B}_j\mathbf{G}_{0j}^{-1}\mathbf{B}_{3j}$$

$$\mathbf{L}_{2j} \equiv \mathbf{L}_j\mathbf{G}_{0j}^{-1}, \quad \mathbf{N}_{2j} \equiv \mathbf{N}_j^2 - \frac{1}{3}\mathbf{B}_j\mathbf{G}_{0j}^{-1}, \quad \mathbf{S}_{3j} \equiv \left[\mathbf{L}_j\mathbf{N}_j^T - \frac{1}{3}(\mathbf{A}_{3j} + \mathbf{N}_j\mathbf{B}_j)\right]\mathbf{G}_{0j}^{-1}$$

$$\mathbf{M}_{4j} \equiv \mathbf{S}_{3j}\mathbf{G}_{0j}\mathbf{N}_j^T - \frac{1}{3}(\mathbf{A}_{4j} + \mathbf{N}_j\mathbf{B}_{3j}), \quad \mathbf{R}_{4j} \equiv \left(\mathbf{M}_{4j} - \frac{1}{3}\mathbf{L}_{2j}\mathbf{B}_j\right)\mathbf{G}_{0j}^{-1}$$

$$\mathbf{Q}_{5j} \equiv \mathbf{R}_{4j}\mathbf{G}_{0j}\mathbf{N}_j^T - \frac{1}{3}\mathbf{L}_{2j}\mathbf{B}_{3j}, \quad \mathbf{P}_{5j} \equiv \left[\mathbf{Q}_{5j} - \frac{1}{3}\mathbf{S}_{3j}\mathbf{B}_j + \frac{1}{5}(\mathbf{A}_{5j} + \mathbf{N}_j\mathbf{B}_{4j})\right]\mathbf{G}_{0j}^{-1}$$

$$\mathbf{R}_{3j} \equiv \mathbf{S}_{3j} + \mathbf{N}_{2j}\mathbf{N}_j - \mathbf{L}_j\mathbf{N}_j^T\mathbf{G}_{0j}^{-1}, \quad \mathbf{T}_{4j} \equiv \mathbf{R}_{3j}\mathbf{N}_j + \mathbf{N}_j\mathbf{R}_{3j} - \mathbf{N}_j\mathbf{N}_{2j}\mathbf{N}_j + \left(\frac{1}{5}\mathbf{B}_{4j} + \frac{1}{9}\mathbf{B}_j\mathbf{G}_{0j}^{-1}\mathbf{B}_j\right)\mathbf{G}_{0j}^{-1}$$

$$\mathbf{Y}_{5j} \equiv \mathbf{T}_{4j}\mathbf{N}_j + \left[\frac{1}{5}(\mathbf{A}_{5j} + \mathbf{N}_j\mathbf{B}_{4j}) - \frac{1}{3}(\mathbf{N}_{2j}\mathbf{A}_{3j} + \mathbf{R}_{3j}\mathbf{B}_j)\right]\mathbf{G}_{0j}^{-1}$$

Note some structural features of the impedance boundary conditions: transposition of the operators on the left-hand sides of equalities (4.2) and (4.3) and the increase in the degree of the wave operator for $s = 2, 4, \ldots$. For $s = 3, 5, \ldots$ the correction contains derivatives with respect to the longitudinal coordinate of the operators of the previous iteration. The operators of the impedance boundary conditions include both the stiffness matrices of the material and the compliance matrices in the transverse direction.

The quasi-static impedance boundary conditions of orders $s = 0$ and $s = 1$ do not differ from those obtained by other researchers and by other methods.[5-14,23-25] We will now compare the propagator matrices obtained using impedance boundary conditions (4.1)–(4.3) and using the method of expanding the propagator matrix of the partial waves $\exp[i(\mathbf{kX} - \omega T)]$ with respect to the wave number, taking into account the components of higher orders.[21,29,30] We will express the normal derivatives of $\mathbf{d}$ and $\mathbf{t}_z$ from the equations of elasticity and Hooke's law in dimensional form $\partial_T \to i\omega$, $\partial_{X_\alpha} \to ik_\alpha$

$$\partial_z \boldsymbol{\xi} = \mathbf{A}_* \boldsymbol{\xi}, \quad \boldsymbol{\xi} \equiv \begin{bmatrix} \mathbf{d} \\ \mathbf{t}_z \end{bmatrix}, \quad \mathbf{A}_* \equiv \begin{bmatrix} -\mathbf{N}_j^T & \mathbf{G}_{0j}^{-1} \\ \mathbf{L}_j & -\mathbf{N}_j \end{bmatrix}$$

and consider the impedance boundary conditions, beginning with $s = 2$. From relations (4.2) and (4.3) we express the values of ξ on the faces and obtain an approximation $\mathbf{B}_{ii}$ of the exact propagator matrix $\mathbf{B}(h) = e^{2h\mathbf{A}_*}$ for a single layer $j = 1$

$$\mathbf{A}_{II}^+ \boldsymbol{\xi}^+ = \mathbf{A}_{II}^- \boldsymbol{\xi}^-, \quad \mathbf{A}_{II}^\pm(h) \equiv 1 \mp h\mathbf{A}_* - h^2 \mathbf{O}_2 \mp h^3 \mathbf{O}_3$$

$$\boldsymbol{\xi}^+ = \mathbf{B}_{II} \boldsymbol{\xi}^- \to \mathbf{B}_{II} \equiv \left(\mathbf{A}_{II}^+\right)^{-1} \mathbf{A}_{II}^- = 1 + 2h\mathbf{A}_* + O\left(h^2\right)$$

$$\mathbf{O}_2 \equiv \begin{bmatrix} 0 & \mathbf{G}_{0j}^{-1}\mathbf{N}_j \\ 0 & 0 \end{bmatrix}, \quad \mathbf{O}_3 \equiv \begin{bmatrix} 0 & \mathbf{G}_{0j}^{-1}\mathbf{N}_{2j} \\ 0 & 0 \end{bmatrix}$$

The approximate matrix $\mathbf{B}_{ii}$ contains a section of a power series for $\mathbf{B}(h)$, and the last iterations refine this approximation. Unlike the approximation for $\mathbf{B}_{ii}$, obtained earlier,[20,21,29,30] the approximation proposed here takes into account the non-zero matrices $\mathbf{O}_2$ and $\mathbf{O}_3$, which provides the same asymptotic accuracy of the displacements and stresses.

5. The case of a dissipative material

When deriving the impedance boundary conditions we assumed that the materials of the layers are elastic. We will now assume that energy dissipation is possible in the layers, for example, in accordance with the Kelvin–Voight model $\sigma_{pj} = \left\{ g'^j_{pq} + g''^j_{pq} \partial_t \right\} \varepsilon_{qj}$, where $||g'^j_{pq}||$ is the tensor of elastic constants, while $||g''^j_{pq}||$ is the tensor of viscosity coefficients. The latter is assumed to be small with relaxation times $\tau^j_{pq} = g''^j_{pq} / g'^j_{pq} \ll 1$ and, in the case of a time dependence in the form $e^{-i\omega t}$

$$g^j_{pg} = g'^j_{pq} - i\omega g''^j_{pq} = g'^j_{pq}\left(1 - i\omega\tau^j_{pq}\right) = g'^j_{pq}\left(1 - i2\pi\varepsilon c_0 \tau^j_{pq} / h\right) = O(1)$$

up to frequencies of the order of $\omega\tau^j_{pq} = 2\pi\tau^j_{pq}/T_0 = O(1)$, and also somewhat longer relaxation times $\tau^j_{pq} \sim L/c_0$. This does not change the asymptotic derivation procedure, and in formulae (2.4)–(2.6) and (4.1)–(4.3) we can put $\mathbf{G}_j = \mathbf{G}_j(\omega)$, $\sigma_{pj} = g^j_{pq}\varepsilon_{qi}$. Strictly speaking, for small ω, comparable with ε, on the right-hand side of formulae (1.3) and (2.4) we must drop higher-order terms, which appear due to the viscosity. Retaining these terms, we simplify the derivation and we sum all the additional components of the corresponding asymptotic series into compact expressions with a matrix of the complex moduli $\mathbf{G}_j(\omega)$.

We will assume, for example, that for one layer ($j = 1$) the stiffness matrix has the following expansion in powers of ε

$$\mathbf{G}_j = \mathbf{G}_j^0 + \varepsilon \mathbf{G}_j^1 + \varepsilon^2 \mathbf{G}_j^2 + \ldots \tag{5.1}$$

Then the dimensionless recurrence relations (1.4) and (1.5) are replaced as follows:

$$\mathbf{t}_{zj}^s = \left(\mathbf{G}_{\|j}^0\right)^T \mathbf{D}^T \mathbf{d}_j^{s-1} + \left(\mathbf{G}_{\|j}^1\right)^T \mathbf{D}^T \mathbf{d}_j^{s-2} + \ldots + \mathbf{G}_{0j}^0 \partial_z \mathbf{d}_j^s + \mathbf{G}_{0j}^1 \partial_z \mathbf{d}_j^{s-1} + \ldots$$

$$\mathbf{G}_{0j}^0 \partial_z^2 \mathbf{d}_j^s + \mathbf{G}_{0j}^1 \partial_z^2 \mathbf{d}_j^{s-1} + \ldots = -\mathbf{D}_{1j}^0 \partial_z \mathbf{d}_j^{s-1} - \mathbf{D}_{1j}^1 \partial_z \mathbf{d}_j^{s-2} - \ldots + \mathbf{A}_j^0 \mathbf{d}_j^{s-2} + \mathbf{A}_j^1 \mathbf{d}_j^{s-3} + \ldots$$

Relations (2.1) are not changed, while (2.2) and (2.3) take the form

$$\mathbf{t}_+^1 = \mathbf{G}_{0j}^0 \mathbf{d}_-^1 + \mathbf{G}_{0j}^1 \mathbf{d}_-^0 + \left(\mathbf{G}_{\|j}^0\right)^T \mathbf{D}^T \mathbf{d}_+^0, \quad \mathbf{t}_-^1 = 0 - \mathbf{D}\mathbf{G}_{\|j}^0 \mathbf{d}_-^0 = 0 - \mathbf{N}_j^0 \mathbf{t}_+^0 \tag{5.2}$$

$$\mathbf{t}_+^2 = \mathbf{G}_{0j}^0 \mathbf{d}_-^2 + \mathbf{G}_{0j}^1 \mathbf{d}_-^1 + \mathbf{G}_{0j}^2 \mathbf{d}_-^0 + \left(\mathbf{G}_{\|j}^0\right)^T \mathbf{D}^T \mathbf{d}_+^1 + \left(\mathbf{G}_{\|j}^1\right)^T \mathbf{D}^T \mathbf{d}_+^0 + \frac{1}{3}\mathbf{B}_j^0 \mathbf{d}_-^0$$

$$\mathbf{t}_-^2 = 0 - \mathbf{D}\mathbf{G}_{\|j}^0 \mathbf{d}_-^1 - \mathbf{D}\mathbf{G}_{\|j}^1 \mathbf{d}_-^0 + \mathbf{A}_j^0 \mathbf{d}_+^0 \tag{5.3}$$

Summing the sections of the series

$$\mathbf{t}_\mp = \mathbf{t}_\mp^0 + \varepsilon \mathbf{t}_\mp^1 + \varepsilon^2 \mathbf{t}_\mp^1 \quad \text{and} \quad \mathbf{d}_\mp = \mathbf{d}_\mp^0 + \varepsilon \mathbf{d}_\mp^1 + \varepsilon^2 \mathbf{d}_\mp^1$$

we obtain from relations (5.2) and (5.3)

$$\mathbf{t}_- = 0 - \varepsilon \mathbf{N}_j \mathbf{G}_{0j} \mathbf{d}_- + \varepsilon^2 \mathbf{A}_j \mathbf{d}_+ + O(\varepsilon^3), \quad \mathbf{t}_+ = \mathbf{G}_{0j} \mathbf{d}_- + \varepsilon \mathbf{G}_{0j} \mathbf{N}_j^T \mathbf{d}_+ + \frac{1}{3}\varepsilon^2 \mathbf{B}_j \mathbf{d}_- + O(\varepsilon^3)$$

which, in dimensional variables, gives a difference $O(\varepsilon^3)$ from the impedance boundary conditions (4.1) for $s = 0, 1, 2$. The consideration of higher orders and layered packets is similar.

In the final impedance boundary conditions, derived with an error $O(\varepsilon^n)$, due to the matrix $\mathbf{G}_j(\omega)$ terms of the order of the same error are added. The derivation procedure also does not break down in the case of a non-linear relation $\mathbf{G}_j(\omega)$ if expansion (5.1) occurs; an important difference arises when the “viscous” and “elastic” components are comparable.

In particular, the non-linear relation $\mathbf{G}_j(\omega)$ may correspond to such a non-traditional material as a liquid-crystalline nematic elastomer, which is of interest both for applications in biomechanics and medicine, and as a possible matrix for nanocomposites. A review of the literature on the properties of nematic materials can be found in Refs 31 and 32. For these, the presence of viscosity and additional internal degrees of freedom with respect to relative rotation around the main axis of orientation of the molecules **n** (the “director”), situated, to be specific, in the direction of the x_3 axis, is characteristic. In the low-frequency region for other constant physical conditions (temperature, pressure, etc.) the material is characterized by eleven constants: the density, five independent components of the matrix $\mathbf{G}_j'$ (similar to transversal isotropy with anisotropy axis **n**) for elastic processes and one relaxation time τ_R, and two stiffnesses with respect to rotation's D_1 and D_2 with relaxation times τ_1 and τ_2. The equations of rotation can be assumed to be quasi-static.[32] For the purposes of this paper, it is important that, as was shown in Refs 33 and 34, this type of elastomer can be simply described as a compressed viscoelastic material with a renormalized matrix $\mathbf{G}_j(\omega)$ with non-zero elements

$$g_{pq} = (1 - i\omega\tau_R) g'_{pq} \quad (pq = 11, 12, 13, 33, 66), \quad g_{11} = g_{22},$$

$$g_{13} = g_{23}, \quad g_{44} = g_{55} = (1 - i\omega\tau_R) g_{44}^R$$

where

$$g_{44}^R(\omega) = g'_{44} - \frac{D_2^2}{4D_1} \frac{(1 - i\omega\tau_2)^2}{(1 - i\omega\tau_1)(1 - i\omega\tau_R)} = g'_{44} - \frac{D_2^2}{4D_1}\left[1 + i\omega(\tau_1 + \tau_R - 2\tau_2) + O(\omega^2)\right] \tag{5.4}$$

Hence, the impedance boundary conditions can also be used to describe a nematic lining or coating on the assumption that $\omega \max(\tau_1, \tau_2, \tau_R) \ll 1$.

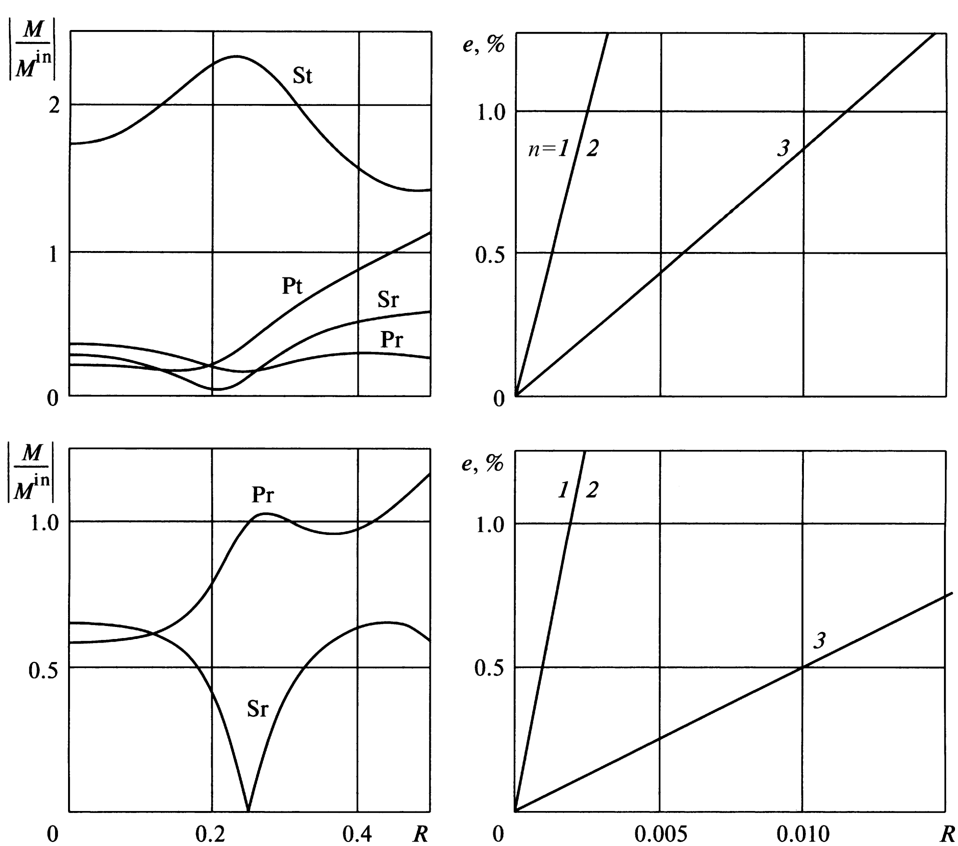

Fig. 1.

6. Results of numerical testing of the impedance boundary conditions

Since, when investigating eigenwaves and spectra, partial waves are the main tool, we will compare the amplitudes of the partial waves obtained using the exact matrix method[15-19] and the asymptotic impedance boundary conditions. We will consider the lining between two isotropic half-spaces. Suppose a plane harmonic P-wave or an S-wave

$$\mathbf{U}^{\text{in}} = \mathbf{U}^{\text{in}}_{P,S}\exp\left[ik_{P,S}\left(x_1\cos\theta^{\text{in}} - x_3\sin\theta^{\text{in}}\right) - i\omega t\right].$$

with amplitude M^{in} is incident from the upper half-space (in which all the quantities are given a plus superscript) on to the interface of the media at a certain angle θ^{in}. Then, in the upper half-space we obtain reflected P- and S-waves (Pr and Sr), in the lower half-space (where quantities are given a minus subscript) we obtain the two transmitted waves (Pt and St), and in each layer there are direct and reflected waves with a propagation direction given by Snell's law. There are corresponding calculations (see, for example, Ref. 19), but they are omitted here. In the approximate solution the incident and reflected waves in the upper half-space ($Z=Z^+$) and in the lower half-space ($Z=Z^-$) are related by approximate impedance boundary conditions (2.4). We will take as the accuracy criterion the root mean square of the relative error

$$e = \left\{\frac{1}{4}\sum_{[+,-;P,S]}\left|1 - \frac{M^{\text{as}}}{M^{\text{ex}}}\right|^2\right\}^{1/2} \times 100\%$$

where M^{ex} is the complex amplitude of the corresponding exact solution, while M^{as} is the amplitude of the approximate solution when summation is carried out over all waves in the half-spaces.

We can similarly consider the upper half-space with the coating. For a stress-free coating surface we use impedance boundary conditions (2.5) in the approximate calculation. A comparison was carried out for the reflected Pr and Sr waves. For a fixed face coating we use the impedance boundary conditions (2.6) in the approximate calculation.

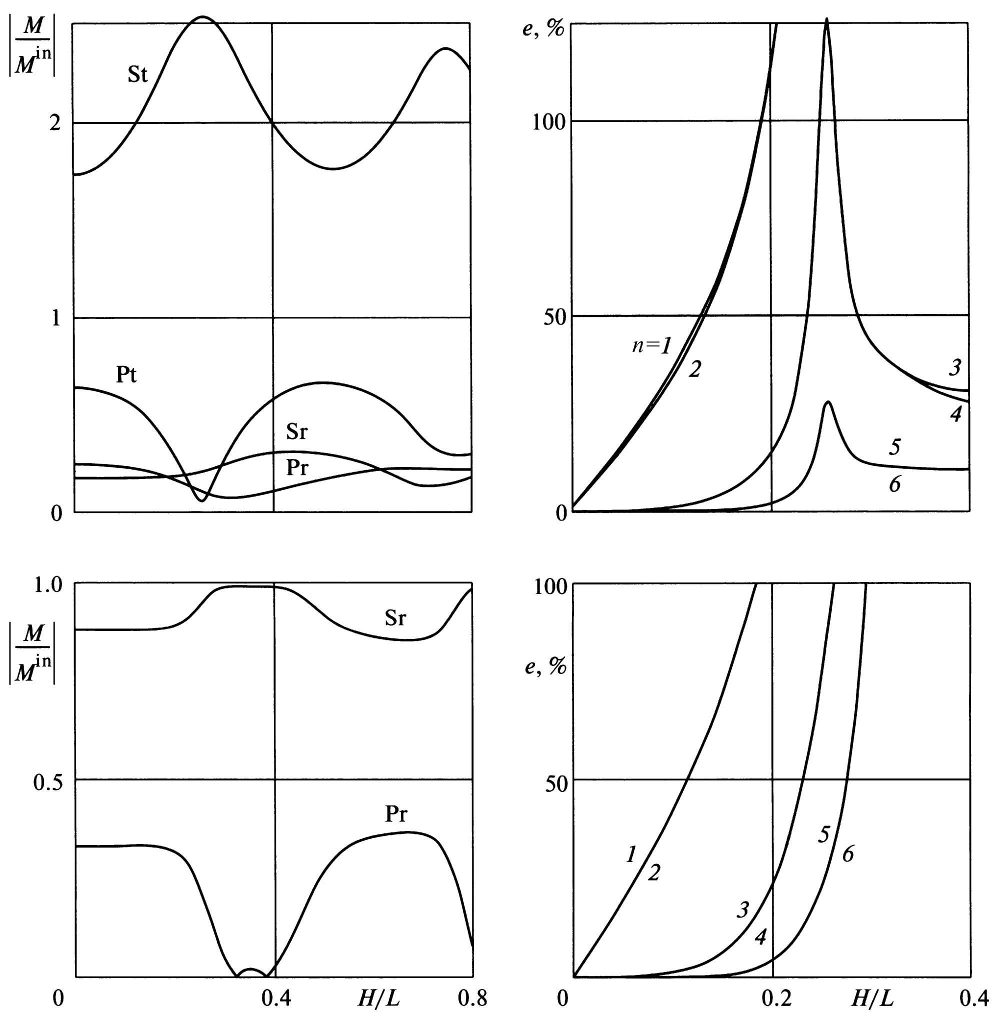

Fig. 2.

In the calculations we took the following typical parameters for metals (Al)

$$\rho = 2700\ \text{kg/m}^3, \quad E = 6.886 \cdot 10^{10}\ \text{N/m}^2, \quad \nu = 0.3442$$

isotropic plastics (Po)

$$\rho = 1060\ \text{kg/m}^3, \quad E = 3.764 \cdot 10^{9}\ \text{N/m}^2, \quad \nu = 0.3425$$

and orthotropic plexiglass (Eg)

$$\rho = 2000\ \text{kg/m}^3, \quad g_{11} = 4.83 \cdot 10^{10}, \quad g_{13} = 0.57 \cdot 10^{10},$$

$$g_{33} = 1.486 \cdot 10^{10}, \quad g_{55} = 0.44 \cdot 10^{10}\ \text{N/m}^2$$

The cross-ply packet consists of two orthotropic layers. We varied the isotropic materials in the substrate, the type of incident waves and the angle of incidence.

In Fig. 1 we show the dimensionless amplitudes of the exact solution (the upper part of Fig. 1) and the relative error e (the lower part of Fig. 1), for the incidence of an S-wave at an angle of 70° to the interface of the upper medium Al and the lower medium Po with a lining of two layers of Eg (cross-ply with a rotation of the principal orthotropy axes of 0° and 90° respectively; $H_1 = H_2$). Along the horizontal axis we show the scale $R = H_1/L_1 + H_2/L_2$, where L_1 and L_2 are the least of the characteristic wavelengths in each layer. The graphs in the lower part of Fig. 1 are given for a medium of Al with the same coating with a free boundary. The number's on the curves n correspond to the asymptotic error $O(\varepsilon^n)$. For an error $e \le 1\%$ the value of R is greatest for the impedance boundary conditions of a rigidly clamped face and least for the impedance boundary conditions of a free surface (2.5); for the impedance boundary conditions of the lining (2.4) this value is intermediate between the two. For quasistatic impedance boundary conditions, taking into account terms with $n = 0$ and 1, the width of the interval of applicability varies in the range from 1/500 to 1/200, and for $n = 2$ it increases, varying in the range from 1/100 to 1/20.

When testing the impedance boundary conditions for a single layer we varied the orientation of the principal orthotropy axes of the layer with respect to the coordinate axes. Typical graphs of exact values of the amplitudes and relative error e with an indication of the orientation of the main anisotropy axis 1 with respect to the x_1 axis are shown in the upper part of Fig. 2. They correspond to the incidence of an S-wave at an angle of 80° to the interface of the Al and Po media with a lining of Eg at an angle of orientation of the axes of 0°. The graphs in the lower part of Fig. 2 are for Al with the same coating and with a free boundary.

The width of the interval of applicability, which gives an error of no greater than 1%, increases sharply for a single layer. For $n = 1$ and $n = 2$ it ranges from 1/50 to 1/30, and increases to 1/10 for $n = 3$ and $n = 4$, and to 1/6 for $n = 5$ and $n = 6$. As stated above, the impedance boundary condition of the lining (4.1) gives an intermediate value of the width of the interval of applicability between the values in the case of the impedance boundary conditions of a rigid clamping (4.3) and a free surface (4.2). It follows directly from formulae (2.4)–(2.6) for a packet that they are converted into impedance boundary conditions (4.1)–(4.3) in cases when the materials of the layers are the same, or when the thickness of each layer in the packet, apart from one, approaches zero. This fact can also be seen in the numerical calculations, if,

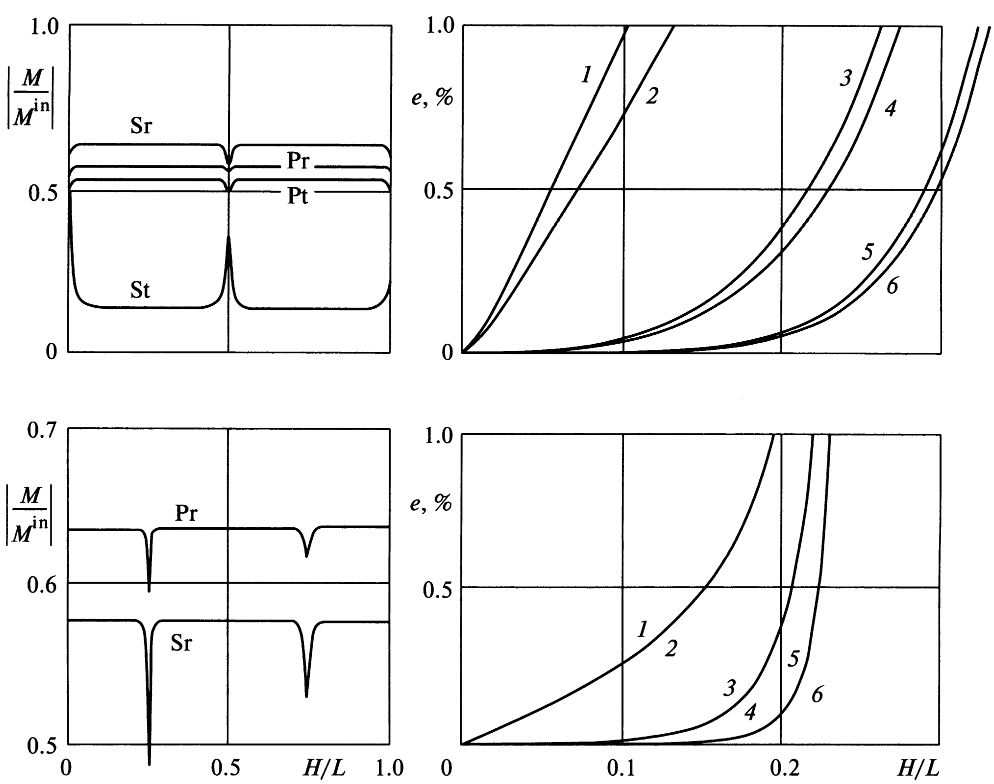

Fig. 3.

for example, one of the layers is much thicker than the others. Moreover, impedance boundary conditions (2.4)–(2.6) and (4.1)–(4.3) do not distort the balance of the power flows, averaged over a period, i.e., when the waves pass through the lining or the coating, no absorption or dissipation of energy occurs. Hence, the impedance boundary conditions obtained also describe wave effects in the case of the same materials, and when the stiffness of the lining or the coating increases, which is also confirmed by numerical tests. Of course, it is not possible to obtain the limit case of the equations of the oscillations of a stiff plate in a soft medium, by this way.

For a single-layer coating and lining of nematic material (Ne) we took the following values of the parameters:[31-33]

$$\tau_1 = 0.01\ \mathrm{s},\quad \tau_2 = 5\cdot 10^{-5}\ \mathrm{s},\quad \tau_R = 10^{-6}\ \mathrm{s},\quad \rho = 10^3\ \mathrm{kg/m^3},$$

$$g'_{11} \approx g'_{33} = 2.00\ \mathrm{GPa},\quad g'_{12} \approx g'_{13} = 1.999\ \mathrm{GPa},$$

$$g'_{44} = 0.133\ \mathrm{GPa},\quad g'_{66} = 0.1\ \mathrm{GPa},\quad D_2^2\big/(4D_1) = 0.131\ \mathrm{MPa}$$

(or $D_2^2/(4D_1) = g'_{44}$, which is called an ideal case, since, according to relations (5.4) $g_{44}^R(0) = 0$, ... it corresponds more to the behaviour of a liquid crystal). The values $g'_{11}, g'_{12}, g'_{13}, g'_{33}$ are close due to the high bulk compression modulus. The anisotropy axis was oriented along each of the coordinate axes alternately, and the materials of the substrate were the same as before. Typical graphs of the amplitudes and relative error are shown in the upper part of Fig. 3 for the case of an S-wave, incident at an angle of 70° on the interface of Al and Po with a lining of Ne with an angle of orientation of the principal axes of 0°. The graphs in the lower part of Fig. 3 are for Al with the same coating and with a free boundary (a non-ideal nematic material).

The error e changes only slightly depending on the orientation of the principal axes of the nematic medium, but for all the impedance boundary conditions the range of applicability is greater than for the case of an ideally elastic material, which may be due to the effect of viscosity. Note that the iteration $s=2$ may be more accurate than the later ones, but this effect occurs in the frequency range when the approximate description of the nematic medium, proposed earlier in Refs 33 and 34, is not valid.

7. Conclusions

We have constructed an approximate asymptotic model of elastic coatings and linings of high order of accuracy for elastic and viscoelastic materials. Its main result is the possibility of replacing the action of the coating or lining to a thick solid by formulating effective impedance boundary conditions at the interfaces. Physically this can be explained by the absence at low frequencies of propagating modes in the layer with "restricted" conditions on the faces, i.e., when at least one of the displacements vanishes on at least one of the faces. The results for the low-order model are identical with the results of other researchers. The presence of higher orders is important, since the first orders only give a quasi-static approximation. The sixth-order model is low-frequency, but is possibly not the long-wave model. The accuracy of the impedance boundary conditions for laminated packets is much lower than the accuracy for a single-layer coating (lining) and recalls the classical theory of Kirchhoff plates. For one layer the accuracy of the impedance boundary conditions, even for the second iteration (due to the occurrence in the relations of wave operators) is similar to the accuracy of the results of the Timoshenko–Reisner theory of plates; here the model itself is relatively compact. The limit of the range of applicability of the impedance boundary conditions for this case is the first frequency of natural quasi-resonance. The next iterations look more cumbersome, but the range of applicability is increased. The method enables one to construct a model of any asymptotic accuracy, but each step in the recurrence procedure increases the accuracy by one order. This is less than in the theory of plates with monoclinic anisotropy or in the impedance boundary conditions for a liquid lining, where this step comprises two orders.[41,42] Another characteristic feature of the model is its qualitative improvement for even iteration indices (since, in this case the wave operators are raised to a power), whereas odd iterations correct the result of the previous step by additional differentiation with respect to the longitudinal coordinates.

Results similar to those derived for conditions of complete interface contact and a free or rigidly clamped face may be obtained for impedance boundary conditions involving slippage, friction, etc.

Acknowledgments

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